8 Matrice. Operacije sa matricama.

Zadatak 8.1 *Date su matrice*

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix}.$$

Izračunati matrice $A + B, A - B, A \cdot B$.

Rješenje: Matrice A i B su formata 3×3 pa su moguće tražene operacije nad njima.

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 11 \\ 6 & 2 & 10 \\ 6 & 3 & 17 \end{bmatrix} \\ A - B &= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -1 \\ 0 & 2 & 2 \\ -4 & -1 & -3 \end{bmatrix} \\ A \cdot B &= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 39 & 8 & 78 \\ 39 & 9 & 86 \\ 39 & 13 & 80 \end{bmatrix}. \end{aligned}$$

Zadatak 8.2 *Data je matrica*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Izračunati $P(A) = 3A^2 - 5A - 2E$, gdje je E jedinična matrica.

Rješenje:

$$\begin{aligned} P(A) &= 3A^2 - 5A - 2E = \\ &= 3\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \\ &= 3\begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 21 & 12 \\ 18 & 21 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -2 \\ 3 & 14 \end{bmatrix}. \end{aligned}$$

Zadatak 8.3 *Dat je polinom $P(x) = 2x - x^{-2} - 3$ i matrica*

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Izračunati $P(A)$.

Rješenje:
Jasno je da vrijedi $P(A) = 2A - A^{-2} - 3E$, gdje je E jedinična matrica. Prvo ćemo izračunati inverznu matricu matrice A po formuli

$$A^{-1} = \frac{1}{\det A} \cdot adj A.$$

Imamo

$$\det A = \begin{vmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{vmatrix} \begin{matrix} 4 & -1 \\ 0 & -1 \\ 0 & 0 \end{matrix} = -8.$$

Kofaktori matrice A su:

$$A_{11} = \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = -2 \quad A_{12} = -\begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} = 0 \quad A_{13} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix} = 2 \quad A_{22} = \begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix} = 8 \quad A_{23} = -\begin{vmatrix} 4 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} -1 & -2 \\ -1 & 3 \end{vmatrix} = -5 \quad A_{32} = -\begin{vmatrix} 4 & -2 \\ 0 & 3 \end{vmatrix} = -12 \quad A_{33} = \begin{vmatrix} 4 & -1 \\ 0 & 1 \end{vmatrix} = -4$$

pa je adjungovana matrica matrice A

$$adj A = \begin{bmatrix} -2 & 2 & -5 \\ 0 & 8 & -12 \\ 0 & 0 & -4 \end{bmatrix}.$$

Sada je

$$A^{-1} = -\frac{1}{8}\begin{bmatrix} -2 & 2 & -5 \\ 0 & 8 & -12 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

i

$$\begin{aligned} A^{-2} &= A^{-1} \cdot A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{32} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

Konačno, imamo da je

$$
\begin{aligned}
P(A) &= 2A - A^{-2} - 3E = \\
&= 2\begin{bmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{32} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} 8 & -2 & -4 \\ 0 & -2 & 6 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{32} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \\
&= \begin{bmatrix} \frac{79}{16} & -\frac{35}{16} & -\frac{131}{32} \\ 0 & -6 & \frac{27}{4} \\ 0 & 0 & \frac{3}{4} \end{bmatrix}.
\end{aligned}
$$

Zadatak 8.4 *Dat je polinom* $P(x) = -2 + 3x + x^{-2}$ *i matrica*

$$
A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}
$$

Izračunati $P(A)$.

Rješenje:
Jasno je da vrijedi $P(A) = -2E + 3A + A^{-2}$, gdje je E jedninična matrica. Prvo ćemo izračunati inverznu matricu matrice A po formuli

$$
A^{-1} = \frac{1}{\det A} \cdot adj A.
$$

Imamo

$$
\det A = \begin{vmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{vmatrix}\begin{matrix} 1 & -1 \\ 0 & 2 \\ 0 & 0 \end{matrix} = -6.
$$

Kofaktori matrice A su:

$$
\begin{aligned}
A_{11} &= \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = -6 & A_{12} &= -\begin{vmatrix} 0 & 1 \\ 0 & -3 \end{vmatrix} = 0 & A_{13} &= \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{21} &= -\begin{vmatrix} -1 & -2 \\ 0 & -3 \end{vmatrix} = -3 & A_{22} &= \begin{vmatrix} 1 & -2 \\ 0 & -3 \end{vmatrix} = -3 & A_{23} &= -\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{31} &= \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = 3 & A_{32} &= -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = -1 & A_{33} &= \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2
\end{aligned}
$$

pa je adjungovana matrica matrice A

$$
adj A = \begin{bmatrix} -6 & -3 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{bmatrix}.
$$

Sada je

$$
A^{-1} = -\frac{1}{6}\begin{bmatrix} -6 & -3 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}
$$

i

$$
\begin{aligned}
A^{-2} &= A^{-1} \cdot A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \\
&= \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{36} \\ 0 & 0 & \frac{1}{9} \end{bmatrix}.
\end{aligned}
$$

Konačno, imamo da je

$$
\begin{aligned}
P(A) &= -2E + 3A + A^{-2} = \\
&= -2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{36} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \\
&= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & -3 & -6 \\ 0 & 6 & 3 \\ 0 & 0 & -9 \end{bmatrix} + \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{36} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \\
&= \begin{bmatrix} 2 & -\frac{9}{4} & -\frac{25}{4} \\ 0 & \frac{17}{4} & \frac{109}{36} \\ 0 & 0 & -\frac{98}{9} \end{bmatrix}.
\end{aligned}
$$

Zadatak 8.5 *Dokazati da za matricu*

$$
A(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}
$$

vrijedi $A(t) \cdot A(r) = A(t + r), t, r \in \mathbb{R}$.

Rješenje:

Kako je

$$A(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \quad i \quad A(x) = \begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix}$$

tada je

$$\begin{aligned} A(t) \cdot A(r) &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix} = \\ &= \begin{bmatrix} \cos t \cdot \cos r - \sin t \cdot \sin r & -\cos t \cdot \sin r - \sin t \cdot \cos r \\ \sin t \cdot \cos r + \cos t \cdot \sin r & -\sin t \cdot \sin r + \cos t \cdot \cos r \end{bmatrix} = \\ &= \begin{bmatrix} \cos t \cdot \cos r - \sin t \cdot \sin r & -(\cos t \cdot \sin r + \sin t \cdot \cos r) \\ \sin t \cdot \cos r + \cos t \cdot \sin r & \cos t \cdot \cos r - \sin t \cdot \sin r \end{bmatrix} = \\ &= \begin{bmatrix} \cos(t+r) & -\sin(t+r) \\ \sin(t+r) & \cos(t+r) \end{bmatrix} = A(t+r). \end{aligned}$$

Zadatak 8.6 *Date su matrice*

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Izračunati matrice $P = AB^{-1} + A^{-2} + 2E$ *i* $Q = 2A - 3B^{-1} + A^{-1}B$, *gdje je* E *jedinična matrica.*

Rješenje:
Izračunajmo inverznu matricu matrice A po formuli

$$A^{-1} = \frac{1}{\det A} \cdot adjA.$$

Imamo

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}\begin{matrix} 2 & 0 \\ 4 & 3 \\ 1 & 1 \end{matrix} = \\ &= (6+0-4) - (-3+4+0) = 2 - 1 = 1. \end{aligned}$$

Kofaktori matrice A su

$$A_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1 \quad A_{12} = -\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = -2 \quad A_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1$$
$$A_{21} = -\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = -1 \quad A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \quad A_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -2$$
$$A_{31} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 \quad A_{32} = -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = -8 \quad A_{33} = \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} = 6$$

pa je adjungovana matrica

$$adjA = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix}$$

Inverzna matrica matrice A je

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix}.$$

Izračunajmo inverznu matricu matrice B po formuli

$$B^{-1} = \frac{1}{\det B} \cdot adjB.$$

Imamo

$$\begin{aligned} \det B &= \begin{vmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix}\begin{matrix} 0 & 1 \\ 4 & 2 \\ 1 & 0 \end{matrix} = \\ &= (0+3+0) - (2+0-4) = 3 + 2 = 5. \end{aligned}$$

Kofaktori matrice B su

$$B_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2 \quad B_{12} = -\begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} = 7 \quad B_{13} = \begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} = -2$$
$$B_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \quad B_{22} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \quad B_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$$
$$B_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \quad B_{32} = -\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 4 \quad B_{33} = \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} = -4$$

pa je adjungovana matrica

$$adjB = \begin{bmatrix} -2 & 1 & 1 \\ 7 & -1 & 4 \\ -2 & 1 & -4 \end{bmatrix}$$

Inverzna matrica matrice B je

$$B^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 1 & 1 \\ 7 & -1 & 4 \\ -2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{1}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \end{bmatrix}.$$

Izračunajmo matricu A^{-2}

$$A^{-2} = A^{-1}\cdot A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -10 & 29 \\ -16 & 27 & -78 \\ 11 & -19 & 55 \end{bmatrix}.$$

Sada je

$$\begin{aligned} P &= AB^{-1} + A^{-2} + 2E = \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{1}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \end{bmatrix} + \begin{bmatrix} 6 & -10 & 29 \\ -16 & 27 & -78 \\ 11 & -19 & 55 \end{bmatrix} + \\ &\quad + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{38}{5} & -\frac{49}{5} & \frac{151}{5} \\ -\frac{71}{5} & \frac{148}{5} & -\frac{382}{5} \\ \frac{58}{5} & -\frac{94}{5} & \frac{286}{5} \end{bmatrix} \end{aligned}$$

i

$$\begin{aligned} Q &= 2A - 3B^{-1} + A^{-1}B = \\ &= 2\cdot \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} - 3\cdot \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{1}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \end{bmatrix} + \\ &\quad + \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & -2 \\ 8 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} -\frac{6}{5} & \frac{3}{5} & \frac{3}{5} \\ \frac{21}{5} & -\frac{3}{5} & \frac{12}{5} \\ -\frac{6}{5} & \frac{3}{5} & -\frac{12}{5} \end{bmatrix} + \begin{bmatrix} -1 & -1 & -5 \\ 4 & 4 & 15 \\ -2 & -3 & -11 \end{bmatrix} = \\ &= \begin{bmatrix} \frac{21}{5} & -\frac{8}{5} & -\frac{38}{5} \\ \frac{39}{5} & \frac{53}{5} & \frac{83}{5} \\ \frac{6}{5} & -\frac{8}{5} & -\frac{33}{5} \end{bmatrix}. \end{aligned}$$

Zadatak 8.7 *Riješiti matričnu jednačinu*

$$X(A+E) = 2A - E$$

gdje je

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

i E jedinična matrica.

Rješenje:
Neka je

$$\begin{aligned} P &= A + E = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \\ Q &= 2A - E = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -7 \end{bmatrix}. \end{aligned}$$

Sada matrična jednačina ima oblik

$$XP = Q$$

koja se rješava na sljedeći način

$$\begin{aligned} XP &= Q \quad / \cdot P^{-1} \\ XPP^{-1} &= QP^{-1} \\ XE &= QP^{-1} \end{aligned}$$

pa je rješenje matrica $X = QP^{-1}$.

Izračunajmo matricu P^{-1} po formuli

$$P^{-1} = \frac{1}{\det P} \cdot adjP$$

Determinanta matrice P je

$$\det P = \begin{vmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -12$$

a kofaktori matrice P su

$$P_{11} = \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} = -6 \quad P_{12} = -\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} = 0 \quad P_{13} = \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} = 0$$

$$P_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = -2 \quad P_{22} = \begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix} = -4 \quad P_{23} = -\begin{vmatrix} 2 & 11 \\ 0 & 0 \end{vmatrix} = 0$$

$$P_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -7 \quad P_{32} = -\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = -2 \quad P_{33} = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 6$$

pa je

$$adjP = \begin{bmatrix} -6 & -2 & -7 \\ 0 & -4 & -2 \\ 0 & 0 & 6 \end{bmatrix}.$$

Dakle,

$$P^{-1} = \frac{1}{-12}\begin{bmatrix} -6 & -2 & -7 \\ 0 & -4 & -2 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{7}{12} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Rješenje matrične jednačine je matrica

$$X = QP^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -7 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{7}{12} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{7}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{bmatrix}.$$

Zadatak 8.8 *Riješiti matričnu jednačinu*

$$X\left(2A - 3E\right) = 2E - A$$

gdje je

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

i E jedinična matrica.

Neka je

$$P = 2A - 3E = \begin{bmatrix} 2 & -2 & -4 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & -9 \end{bmatrix}$$

$$Q = 2E - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Sada matrična jednačina ima oblik

$$XP = Q$$

koja se rješava na sljedeći način

$$\begin{aligned} XP &= Q \quad / \cdot P^{-1} \\ XPP^{-1} &= QP^{-1} \\ XE &= QP^{-1} \end{aligned}$$

pa je rješenje matrica $X = QP^{-1}$.

Izračunajmo matricu P^{-1} po formuli

$$P^{-1} = \frac{1}{\det P} \cdot adjP$$

Determinanta matrice P je

$$\det P = \begin{vmatrix} -1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & -9 \end{vmatrix} = 9$$

a kofaktori matrice P su

$$P_{11} = \begin{vmatrix} 1 & 2 \\ 0 & -9 \end{vmatrix} = -9 \quad P_{12} = -\begin{vmatrix} 0 & 2 \\ 0 & -9 \end{vmatrix} = 0 \quad P_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$P_{21} = -\begin{vmatrix} -2 & -4 \\ 0 & -9 \end{vmatrix} = -18 \quad P_{22} = \begin{vmatrix} -1 & -4 \\ 0 & -9 \end{vmatrix} = 9 \quad P_{23} = -\begin{vmatrix} -1 & -2 \\ 0 & 0 \end{vmatrix} =$$

$$P_{31} = \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} = 0 \quad P_{32} = -\begin{vmatrix} -1 & -4 \\ 0 & 2 \end{vmatrix} = 2 \quad P_{33} = \begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -$$

pa je

$$adjP = \begin{bmatrix} -9 & -18 & 0 \\ 0 & 9 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Dakle,

$$P^{-1} = \frac{1}{9}\begin{bmatrix} -9 & -18 & 0 \\ 0 & 9 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}.$$

Rješenje matrične jednačine je matrica

$$X = QP^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & \frac{1}{9} \\ 0 & 0 & -\frac{5}{9} \end{bmatrix}.$$

Zadatak 8.9 *Riješiti matričnu jednačinu*

$$(A - 2B)\,X = 2A - B + E$$

ako je

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

i E jedinična matrica.

Rješenje:
Neka je

$$\begin{aligned} P &= A - 2B = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 1 & -4 \\ 0 & -2 & -1 \\ 0 & 0 & -9 \end{bmatrix} \\ Q &= 2A - B + E = 2 \cdot \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 2 & -2 & -4 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & -8 \end{bmatrix}. \end{aligned}$$

Sada matrična jednačina ima oblik

$$PX = Q$$

koja se rješava na sljedeći način

$$\begin{aligned} P^{-1} \cdot / \quad PX &= Q \quad / \cdot P^{-1} \\ PP^{-1}X &= P^{-1}Q \\ EX &= P^{-1}Q \end{aligned}$$

pa je rješenje matrica $X = P^{-1}Q$.
Izračunajmo matricu P^{-1} po formuli

$$P^{-1} = \frac{1}{\det P} \cdot adj P$$

Determinanta matrice P je

$$\det P = \begin{vmatrix} -3 & 1 & -4 \\ 0 & -2 & -1 \\ 0 & 0 & -9 \end{vmatrix} = -54$$

a kofaktori matrice P su

$$\begin{array}{lll} P_{11} = \begin{vmatrix} -2 & -1 \\ 0 & -9 \end{vmatrix} = 18 & P_{12} = -\begin{vmatrix} 0 & -1 \\ 0 & -9 \end{vmatrix} = 0 & P_{13} = \begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} = 0 \\ P_{21} = -\begin{vmatrix} 1 & -4 \\ 0 & -9 \end{vmatrix} = 9 & P_{22} = \begin{vmatrix} -3 & -4 \\ 0 & -9 \end{vmatrix} = 27 & P_{23} = -\begin{vmatrix} -3 & 1 \\ 0 & 0 \end{vmatrix} = 0 \\ P_{31} = \begin{vmatrix} 1 & -4 \\ -2 & -1 \end{vmatrix} = -9 & P_{32} = -\begin{vmatrix} -3 & -4 \\ 0 & -1 \end{vmatrix} = -3 & P_{33} = \begin{vmatrix} -3 & 1 \\ 0 & -2 \end{vmatrix} = 6 \end{array}$$

pa je

$$adj P = \begin{bmatrix} 18 & 9 & -9 \\ 0 & 27 & -3 \\ 0 & 0 & 6 \end{bmatrix}.$$

Dakle,

$$P^{-1} = \frac{1}{-54} \begin{bmatrix} 18 & 9 & -9 \\ 0 & 27 & -3 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{18} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}.$$

Rješenje matrične jednačine je matrica

$$\begin{aligned} X &= P^{-1}Q = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{18} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & -8 \end{bmatrix} = \\ &= \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{3}{2} & -\frac{51}{54} \\ 0 & 0 & \frac{8}{9} \end{bmatrix}. \end{aligned}$$

Zadatak 8.10 *Riješiti matričnu jednačinu*

$$AX^{-1} = A - X^{-1}$$

ako je

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Rješenje:
Matrična jednačina

$$AX^{-1} = A - X^{-1}$$

je ekvivalentna jednačini

$$AX = A + E$$

jer

$$\begin{aligned} AX^{-1} &= A - X^{-1} \\ AX^{-1} + X^{-1} &= A \\ (A+E)\,X^{-1} &= A \quad / \cdot X \\ (A+E)\,X^{-1}X &= AX \\ A + E &= AX. \end{aligned}$$

Neka je

$$B = A + E = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Sada matriča jednačina ima oblik

$$AX = B$$

koja se rješava na sljedeći način

$$\begin{aligned} A^{-1} \cdot / \;\; AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B. \end{aligned}$$

Inverzna matrica matrice A je matrica

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

pa je rješenje matrične jednačine matrica

$$X = A^{-1}B = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} \end{bmatrix}.$$

Zadatak 8.11 *Matricu*

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

predstaviti kao zbir jedne simetrične i jedne antisimetrične matrice.

Rješenje:

Kvadratna matrica je simetrična ako su joj elementi simetrični u odnosu na glavnu dijagonalu, tj. ako je $a_{ij} = a_{ji}$ i vrijedi

$$A_S = \frac{1}{2}\left(A + A^T\right).$$

Kvadratna matrica je antisimetrična ako je jednaka svojoj negativnoj transponovanoj matrici i vrijedi

$$A_K = \frac{1}{2}\left(A - A^T\right).$$

Kako je

$$A^T = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

onda je

$$\begin{aligned} A_S &= \frac{1}{2}\left(A + A^T\right) = \frac{1}{2}\left(\begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix} \right) = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \\ A_k &= \frac{1}{2}\left(A - A^T\right) = \frac{1}{2}\left(\begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Zadatak 8.12 *Odrediti rang matrice*

$$A = \begin{bmatrix} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned} A &= \begin{bmatrix} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & 1 & 3 & -1 \\ 4 & -3 & -2 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 7 & -2 & 7 \end{bmatrix} \begin{matrix} V_1 \; prepisana \\ 2 \cdot V_1 + V_2 \\ 4 \cdot V_1 - V_3 \end{matrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 13 & 0 \end{bmatrix} \begin{matrix} V_1 \; prepisana \\ V_2 \; prepisana \\ 7 \cdot V_2 - 3 \cdot V_3 \end{matrix} \end{aligned}$$

Dakle, $rang A = 3$.

Zadatak 8.13 *Odrediti rang matrice*

$$A = \begin{bmatrix} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \\ 3 & -2 & 1 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 1 & -8 & 7 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ 2 \cdot V_1 + V_2 \\ 3 \cdot V_1 + V_3 \end{array} \\
&\sim \begin{bmatrix} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \begin{array}{r} V_1 \; prepisana \\ V_2 \; prepisana \\ V_2 - V_3 \end{array}
\end{aligned}$$

Dakle, $rang A = 2$.

Zadatak 8.14 *Odrediti rang matrice*

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 & 2 \\ -1 & 2 & -3 & -1 & -4 \\ 2 & -1 & 3 & 1 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ V_1 - V_2 \\ V_1 + V_3 \\ 2 \cdot V_1 - V_4 \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & -1 & 3 & 1 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ V_2 \; prepisana \\ V_1 + V_3 \\ V_4 \; prepisana \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & -4 & -4 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ V_2 \; prepisana \\ V_3 \; prepisana \\ V_3 - V_4 \end{array}
\end{aligned}$$

Dakle, $rang A = 4$.

Zadatak 8.15 *Odrediti rang matrice*

$$A = \begin{bmatrix} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 2 & 3 & -4 & 1 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ V_1 + V_2 \\ 2V_1 - V_3 \\ 3V_1 - V_4 \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & 0 & 5 & 25 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ V_2 + 4V_3 \\ V_3 - V_4 \end{array}
\end{aligned}$$

Dakle, $rang A = 3$.

Zadatak 8.16 *Odrediti rang matrice*

$$A = \begin{bmatrix} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & -3 & -1 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & -5 & 7 & 3 & 5 \\ 0 & -1 & 5 & 5 & 9 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ 2V_1 - V_2 \\ 2V_1 + V_3 \\ 3V_1 - V_4 \end{array} \\
&\sim \begin{bmatrix} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & 0 & 18 & 22 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ 5V_2 - V_3 \\ V_2 - V_3 \end{array}
\end{aligned}$$

Dakle, $rang A = 3$.

Zadatak 8.17 *U zavisnosti od realnog parametra λ odrediti rang matrice*

$$A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{bmatrix}
\end{aligned}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda^2 \end{bmatrix} \begin{matrix} V_1 \ prepisana \\ V_2 - V_1 \\ V_3 - \lambda V_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 0 & 2-\lambda-\lambda^2 \end{bmatrix} \begin{matrix} V_1 \ prepisana \\ V_2 \ prepisana \\ V_3 + V_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 0 & (1-\lambda)(2+\lambda) \end{bmatrix}$$

Diskusija:

1. ako je $\lambda = 1$ tada je $rangA = 1$ jer je

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. ako je $\lambda = -2$ tada je $rangA = 2$ jer je

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3. ako je $\lambda \neq 1$ i $\lambda \neq -2$ tada je $rangA = 3$.

Zadatak 8.18 *U zavisnosti od realnog parametra λ odrediti rang matrice*

$$A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 4 & 10 & 1 & \lambda \\ 7 & 17 & 3 & 1 \\ 2 & 4 & 3 & 2 \end{bmatrix}.$$

Rješenje:

$$A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 4 & 10 & 1 & \lambda \\ 7 & 17 & 3 & 1 \\ 2 & 4 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 6 & -15 & \lambda-12 \\ 0 & 10 & -25 & -20 \\ 0 & 2 & -5 & -4 \end{bmatrix} \begin{matrix} V_1 \ prepisana \\ V_2 - 4V_1 \\ V_3 - 7V_1 \\ V_4 - 2V_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 2 & -5 & -4 \\ 0 & 6 & -15 & \lambda-12 \\ 0 & 10 & -25 & -20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 2 & -5 & -4 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} V_1 \ prepisana \\ V_2 \ prepisana \\ V_3 - 3V_2 \\ V_4 - 5V_2 \end{matrix}$$

Diskusija:

1. ako je $\lambda = 0$ tada je $rangA = 2$.
2. ako je $\lambda \neq 0$ tada je $rangA = 3$.